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Three-way decisions method based on matrices approaches oriented dynamic interval-valued information system

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ABSTRACT

Interval-valued Information Systems (IvIS) reflect the uncertain information in real scene, in which the attribute value of objects are all interval values rather than single values. Data information analysed by three-way decisions is not static but dynamically changing in IvIS, which results in the updating of positive region, boundary region and negative region of decision class X. In this paper, a matrix computational framework based on λ-similarity relation is proposed on the variation of the attribute set, attribute value and object set. Based on the framework, some incremental algorithms are proposed to calculate the positive, boundary and negative region of X in dynamic IvIS. Finally, comparative experiments on data sets from UCI are conducted when attribute set, attribute value and object set are updating over time, respectively. Experimental results show that in comparison with the traditional algorithm, the proposed algorithms can effectively save time for the computation of positive, boundary and negative region of X in dynamic IvIS.

1. Introduction

The main idea of three-way decisions (3WD) is to divide the whole into three independent parts and apply different decision methods for different parts, which provides an effective strategy and method for solving complex problems [46,44,47,45,30,35,13]. Various three-way decisions methods derived from Trisecting-Acting-Outcome (TAO) model have been widely applied in machine learning, data mining, pattern recognition and other fields, such as three-way classification [42,23,9,64,61], three-way clustering [52–54,1,51], three-way recommendation [27,59,48,15,58], three-way reduction [19,8,31,38,2,20], three-way strategy [10,37,26], three-way analysis [36,7,33], three-way space [11,12], three-way incremental learning [43,57,29], three-way concept learning [18,14], three-way matroid learning [21,22], three-way group decision-making [25,39,34], etc.

Data analysed by 3WD are commonly explicated and presented through the information system, which includes object set, attribute set and attribute value. However, in the real scene, data information is often not presented in the form of single values, but interval values. In recent years, many scholars have studied Interval-valued Information Systems (IvIS) and have made lots of achievements. By storing the minimum and maximum attribute value of certain object into the neighbour locations of original dataset, Yin et al. [50] transformed the original dataset into the interval-valued dataset. Xie et al. [41] introduced the information structure of IvIS and then proposed new measures of uncertainty for IvIS. Based on the θ-similarity relation, Dai et al. [4] gave θ-accuracy and θ-roughness to evaluate the uncertainty for interval-valued information

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system. Through selecting decision rules with a minimal set of features for classification of IvIS, Yee et al. [49] proposed a classification rule adapting to IvIS based on rough sets. Miao et al. [32] proposed the concept of α-maximal consistent blocks, which can provide simpler discernibility matrices and discernibility functions for IvIS. With the α-maximal consistent blocks, it is faster to get the upper and lower approximations than traditional ways. Liang and Liu [24] explored decision mechanism of interval-valued decision-theoretic rough sets. With a degree of possibility ranking method, decision rules under a certain risk attitude of decision maker was derived. Du and Hu [6] proposed approximate distribution reducts in inconsistent interval-valued ordered decision tables through dominance-based rough set, which was based on substitution of the indiscernibility relation by the dominance relation. With the discernibility matrix, the substitution of the indiscernibility relation by the dominance relation was extracted in IvIS. Liu et al. [28] investigated a new fuzzy relation by means of similarity between interval values in IvIS. By α-approximate equal relation, the unsupervised attribute reduction method based on information entropy was constructed. Based on S-rough sets and grey set, Li and Hu [17] proposed the concept of S-grey rough sets and its lower and upper approximations. S-grey rough sets was proved to be an effective tool for processing the data information in IvIS.

Under the information technology architecture of Internet and Cloud Computing, data in IvIS usually changes dynamically. However, there are few studies on dynamic IvIS, whose attribute sets, object sets and attribute values are not static but rather dynamically changing with time. Zhang et al. [63] proposed an incremental approach for updating lower and upper approximations in the view of λ-similarity relation when an attribute set was added into or removed from the IvIS, respectively. Zhang et al. [62] recommended the average dominance relation and established average dominance rough sets model on interval-valued hesitant fuzzy information system. Then four mechanisms for updating approximations from the perspective of optimism and pessimism were proposed when updating the attribute set. Generally, the approaches above ignore the updating of object sets and attribute value though they are more effective than traditional ways. Aiming at updating attribute sets and object sets effectively, Yu and Xu [55], Yu et al. [56] proposed two dynamic approaches for computing rough approximations in IvIS, respectively. However, these two methods lack a unified framework to deal with dynamic IvIS meanwhile.

In this paper, we establish a unified matrix calculation framework to calculate three-way decisions regions for dynamic IvIS. The rest of the paper is organised as follows. Section 2 reviews some basic knowledge of intervals, IvIS and binary relation S_A^λ. Section 3 proposes 3WD in IvIS based on matrices approach. Section 4 introduces the algorithm of updating matrices for dynamic data sets. Some experimental analysis of matrices approach and traditional approach are discussed in Section 5. Section 6 concludes the paper and elaborates on future studies.

2. Preliminaries

In this section, we introduce the basic knowledge of interval value and similarity relation in IvIS.

Definition 1. [16] Denote $I=\left[b^-,b^+\right]$ given by $\left[b^-,b^+\right]=\left\{x\in\mathbb{R}\mid b^-\le x\le b^+, b^-<b^+\right\}$. I is called interval, where b^- and b^+ are called the lower and upper bounds of I, respectively.

If $b^-\ge 0$, then I is called positive interval and if $b^+\le 0$, then I is called negative interval. When $b^-=b^+$, the interval I will degenerate into a single real number.

Definition 2. Let $I=\left[b^-,b^+\right]$ be an interval, the length of I is defined as follows.

$$l(I)=b^+-b^- \tag{1}$$

Suppose $I_1=\left[b_1^-,b_1^+\right]$ and $I_2=\left[b_2^-,b_2^+\right]$ are two intervals. The union $I_1\cap I_2$ and the intersection $I_1\cup I_2$ are also defined as an interval.

1. $I_1\cap I_2=\begin{cases}\left[max\left(b_1^-,b_2^-\right),min\left(b_1^+,b_2^+\right)\right], & max\left(b_1^-,b_2^-\right)\le min\left(b_1^+,b_2^+\right)\\ \emptyset, & otherwise\end{cases}$
2. $I_1\cup I_2=\left[min\left(b_1^-,b_2^-\right),max\left(b_1^+,b_2^+\right)\right]$

Definition 3. [63] Let $IvIS=(U,A,V,f)$ be an Interval-valued Information System (IvIS). U is a non-empty finite set of objects; $A=\{a_1,a_2,\cdots,a_m\}$ is a non-empty finite set of attributes; $V=\bigcup_{a\in A}V_a$ is a set of intervals. For $\forall a\in A$, $\exists V_a\in V$ under the mapping from $U\times A$ to V, expressed as $f:U\times A\to V$.

Since similarity relation is less strict than indiscernibility relation, many classical Rough Set Theory (RST) based on similarity relation have been applied in IvIS recent years. Many researchers have done lots of works on similarity relation from different perspectives [40,3,28,5]. Liu et al. [28] noted that similarity relation based on similarity degree S_{ij}^k distinguishes the difference between two interval values very well. Therefore, we take the similarity degree S_{ij}^k as the basis for similarity relation in IvIS.

Table 1
An interval-valued information system.

	a_1	a_2	a_3	a_4	a_5
x_1	[0.1,0.6]	[0.4,0.6]	[0.2,0.4]	[0.0,0.7]	[0.2,0.4]
x_2	[0.6,0.8]	[0.4,0.6]	[0.3,0.5]	[0.8,1.0]	[0.5,0.8]
x_3	[0.2,0.4]	[0.0,0.4]	[0.2,0.8]	[0.6,0.7]	[0.1,0.5]
x_4	[0.7,0.8]	[0.2,0.6]	[0.3,0.6]	[0.2,0.5]	[0.4,0.9]
x_5	[0.3,0.4]	[0.2,0.3]	[0.4,0.8]	[0.4,0.6]	[0.4,1.0]
x_6	[0.4,0.9]	[0.1,0.8]	[0.2,0.8]	[0.1,0.6]	[0.5,0.7]
x_7	[0.6,0.7]	[0.3,0.7]	[0.1,0.2]	[0.8,0.8]	[0.2,0.6]
x_8	[0.6,1.0]	[0.7,0.9]	[0.2,0.4]	[0.4,0.8]	[0.4,0.8]
x_9	[0.1,0.2]	[0.1,0.1]	[0.3,0.6]	[0.7,0.9]	[0.3,0.5]

Definition 4. [63] Let (U, A, V, f) be an IvIS. $U = \{x_1, x_2, \cdots, x_i, \cdots, x_n\}$, $A = \{a_1, a_2, \cdots, a_k, \cdots, a_m\}$. The similarity degree between x_i and x_j under the attribute a_k is defined as follows.

$$S_{ij}^k = \frac{l\left(f\left(x_i, a_k\right) \cap f\left(x_j, a_k\right)\right)}{l\left(\left(f x_i, a_k\right) \cup f\left(x_j, a_k\right)\right)} \tag{2}$$

where $i, j = 1, 2, \ldots, n, k = 1, 2, \ldots, m$.

Obviously, the similarity degree S_{ij}^k satisfies the following properties.

1. $0 \le S_{ij}^k \le 1$
2. $S_{ij}^k = 1$ if and only if $f\left(x_i, a_k\right) = f\left(x_j, a_k\right)$
3. $S_{ij}^k = S_{ji}^k$

Definition 5. [63] Let (U, A, V, f) be an IvIS, $x_i, x_j \in U$, and a real number $\lambda \in [0.5, 1]$. The λ-similarity relation with respect to the attribute set A is defined as follows.

$$S_A^\lambda = \left\{\left(x_i, x_j\right) \in U \times U \mid S_{ij}^k \ge \lambda, \forall a_k \in A\right\} \tag{3}$$

It is easy to verify that the similarity relation S_A^λ satisfies reflexivity and symmetry, but not transitivity. Moreover, the lower and upper approximation of decision class X can be derived by S_A^λ.

Definition 6. Let (U, A, V, f) be an IvIS. $U = \{x_1, x_2, \cdots, x_i, \cdots, x_n\}$, $A = \{a_1, a_2, \cdots, a_m\}$, a real number $\lambda \in [0.5, 1]$. Given a decision class $X \subseteq U$, the lower and upper approximation of X are defined as follows.

$$\underline{X}_{S_A^\lambda} = \left\{x_i \in U \mid S_A^\lambda(x_i) \subseteq X\right\} \qquad \overline{X}_{S_A^\lambda} = \left\{x_i \in U \mid \left(S_A^\lambda(x_i) \cap X\right) \neq \emptyset\right\} \tag{4}$$

Definition 7. The lower and upper approximation of X divide the X into three disjoint regions. They are the positive region $POS_{S_A^\lambda}(X)$, the boundary region $BND_{S_A^\lambda}(X)$ and the negative region $NEG_{S_A^\lambda}(X)$.

$$\begin{aligned} POS_{S_A^\lambda}(X) &= \underline{X}_{S_A^\lambda} \\ BND_{S_A^\lambda}(X) &= \overline{X}_{S_A^\lambda} - \underline{X}_{S_A^\lambda} \\ NEG_{S_A^\lambda}(X) &= U - \overline{X}_{S_A^\lambda} \end{aligned} \tag{5}$$

Example 1. In an IvIS given by Table 1, $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, let $X = \{x_1, x_4, x_6\}$. If $\lambda = 0.5$, $A_1 = \{a_1, a_2, a_5\}$, we can obtain the lower and upper approximations of X with respect to A_1 as follows.
$S_{A_1}^{0.5}(x_1) = \{x_1\}$, $S_{A_1}^{0.5}(x_2) = \{x_2, x_4\}$, $S_{A_1}^{0.5}(x_3) = \{x_3\}$, $S_{A_1}^{0.5}(x_4) = \{x_2, x_4\}$, $S_{A_1}^{0.5}(x_5) = \{x_5\}$
$S_{A_1}^{0.5}(x_6) = \{x_6\}$, $S_{A_1}^{0.5}(x_7) = \{x_7\}$, $S_{A_1}^{0.5}(x_8) = \{x_8\}$, $S_{A_1}^{0.5}(x_9) = \{x_9\}$
so we have:

$$\underline{X}_{S_A^\lambda} = \{x_1, x_6\}, \ \overline{X}_{S_A^\lambda} = \{x_1, x_2, x_4, x_6\}$$

and

$$POS_{S_{A_1}^{0.5}}(X) = \{x_1, x_6\}, \ BND_{S_{A_1}^{0.5}}(X) = \{x_2, x_4\}, \ NEG_{S_{A_1}^{0.5}}(X) = \{x_3, x_5, x_7, x_8, x_9\}$$

3. 3WD method based on matrix approach

However, the definitions above are not robust enough for tolerating the noisy samples in real applications. Besides, when the dataset dynamically changing over time, it is time-wasting for interval-valued information system updating the lower and upper approximation of X once again. Refer to the matrices method mentioned in [60], we propose 3WD method based on matrices approaches to obtain positive, boundary and negative region in dynamic IvIS.

Definition 8. Let $U=\{x_1,x_2,\ldots,x_n\}$ and $X\subseteq U$. The characteristic vector $G(X)=(g_1,g_2,\ldots,g_n)^T$ (T denotes the transpose operation) is defined as follows.

$$g_i=\begin{cases}1, & x_i\in X\\ 0, & x_i\notin X\end{cases} \tag{6}$$

where $G(X)$ assigns 1 to an element that belongs to X and 0 to an element that does not belong to X.

Example 2. (Continued from Example 1). Let $U=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9\}$ and $X=\{x_1,x_4,x_6\}$. Then $G(X)=(1,0,0,1,0,1,0,0,0)^T$.

Definition 9. Given an $IvIS=(U,A,V,f)$. $M_{n\times n}^{S_A^\lambda}=\left(m_{ij}\right)_{n\times n}$ is called the relation matrix with respect to the similarity relation S_A^λ. Then m_{ij} is defined as follows.

$$m_{ij}=\begin{cases}1, & \left(x_i,x_j\right)\in S_A^\lambda\\ 0, & \left(x_i,x_j\right)\notin S_A^\lambda\end{cases} \tag{7}$$

Corollary 1. *Let $M_{n\times n}^{S_A^\lambda}=\left(m_{ij}\right)_{n\times n}$ and S_A^λ be a λ-similarity relation, then $m_{ii}=1$ and $m_{ij}=m_{ji}$, $1\le i,j\le n$.*

Example 3. (Continued from Example 1). In the IvIS given by Table 1, $A_1=\{a_1,a_2,a_5\}$, the relation matrix w.r.t. A_1 can be described as follows.

$$M_{9\times 9}^{S_{A_1}^{0.5}}=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&1&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}$$

Definition 10. Let S_A^λ be a λ-similarity relation on U, $\Lambda_{n\times n}^{S_A^\lambda}$ be an induced diagonal matrix of $M_{n\times n}^{S_A^\lambda}=\left(m_{ij}\right)_{n\times n}$.

$$\Lambda_{n\times n}^{S_A^\lambda}=diag\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\cdots,\frac{1}{\lambda_n}\right)=diag\left(\frac{1}{\sum_{j=1}^n m_{1j}},\frac{1}{\sum_{j=1}^n m_{2j}},\cdots,\frac{1}{\sum_{j=1}^n m_{nj}}\right) \tag{8}$$

where $\lambda_i=\sum_{j=1}^n m_{ij}$, $1\le i\le n$.

Corollary 2. $\Lambda_{n\times n}^{S_A^\lambda}=\begin{bmatrix}\frac{1}{\left|S_A^\lambda(x_1)\right|}&0&\cdots&0\\0&\frac{1}{\left|S_A^\lambda(x_2)\right|}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\frac{1}{\left|S_A^\lambda(x_n)\right|}\end{bmatrix}$ *and $1\le\left|S_A^\lambda(x_n)\right|\le n$, $1\le i\le n$.*

Example 4. (Continued from Example 3). The induced diagonal matrix of $M_{9\times 9}^{S_{A_1}^{0.5}}$ can be computed by Definition 10. $\Lambda_{9\times 9}^{S_{A_1}^{0.5}}=diag\,(1,1/2,1,1/2,1,1,1,1,1)$.

Definition 11. [60] Let (U,A,V,f) be an IvIS and $X\subseteq U$. $G(X)$ is the characteristic vector of X. $M_{n\times n}^{S_A^\lambda}$ is relation matrix w.r.t. A and $\Lambda_{n\times n}^{S_A^\lambda}$ is the induced diagonal matrix of $M_{n\times n}^{S_A^\lambda}$. The n-column vector $H(X)$ is defined as follows.

$$H(X) = \Lambda_{n\times n}^{S_A^\lambda} \bullet \left(M_{n\times n}^{S_A^\lambda} \bullet G(X) \right) \tag{9}$$

where $\bullet$ is dot product of matrices.

Corollary 3. *Suppose* $H(X) = (h_1, h_2, \cdots, h_n)^T$. *Let* $\Lambda_{n\times n}^{S_A^\lambda} = diag\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}\right)$, $M_{n\times n}^{S_A^\lambda} = \left(m_{ij}\right)_{n\times n}$, $G(X) = (g_1, g_2, \ldots, g_n)^T$, *where* $\lambda_i = \sum_{j=1}^n m_{ij}$, $1 \le i \le n$. *Then* $\forall i \in \{1, 2, \ldots, n\}$, $h_i = \frac{\sum_{j=1}^n m_{ij} g_j}{\sum_{j=1}^n m_{ij}}$ *and* $0 \le h_i \le 1$.

Definition 12. [60] Let $0 \le \mu \le \nu \le 1$. Four cut matrices of $H(X)$, denoted by $H^{[\mu,\nu]}$, $H^{(\mu,\nu]}$, $H^{[\mu,\nu)}$, and $H^{(\mu,\nu)}$ are defined as follows.

(1)

$$H^{[\mu,\nu]}(X) = (h_i')_{n\times 1}, \quad h_i' = \begin{cases} 1, & \mu \le h_i \le \nu \\ 0, & else \end{cases} \tag{10}$$

(2)

$$H^{(\mu,\nu]}(X) = (h_i')_{n\times 1}, \quad h_i' = \begin{cases} 1, & \mu < h_i \le \nu \\ 0, & else \end{cases} \tag{11}$$

(3)

$$H^{[\mu,\nu)}(X) = (h_i')_{n\times 1}, \quad h_i' = \begin{cases} 1, & \mu \le h_i < \nu \\ 0, & else \end{cases} \tag{12}$$

(4)

$$H^{(\mu,\nu)}(X) = (h_i')_{n\times 1}, \quad h_i' = \begin{cases} 1, & \mu < h_i < \nu \\ 0, & else \end{cases} \tag{13}$$

Theorem 1. *Let* (U, A, V, f) *be an IvIS and* $X \subseteq U$. S_A^λ *be a* λ*-similarity relation and* $H(X) = (h_1, h_2, \cdots, h_n)^T$. *Then the lower and upper approximation of* X *in the IvIS can be derived from the cut matrices as follows.*

(1) The n*-column characteristic vector* $G\left(\underline{X}_{S_A^\lambda}\right)$ *of the lower approximation* $\underline{X}_{S_A^\lambda}$*:*

$$G\left(\underline{X}_{S_A^\lambda}\right) = H^{[1,1]}(X) \tag{14}$$

(2) The n*-column characteristic vector* $G\left(\overline{X}_{S_A^\lambda}\right)$ *of the upper approximation* $\overline{X}_{S_A^\lambda}$*:*

$$G\left(\overline{X}_{S_A^\lambda}\right) = H^{(0,1]}(X) \tag{15}$$

Proof. Suppose $G(X) = (g_1, g_2, \ldots, g_n)^T$, $G\left(\underline{X}_{S_A^\lambda}\right) = \left(g_1', g_2', \ldots, g_n'\right)^T$, $H^{[1,1]}(X) = \left(h_1', h_2', \ldots, h_n'\right)^T$.

(1)$\Rightarrow$: For each $i \in \{1, 2, \cdots, n\}$, if $g_i' = 1$, then $x_i \in \underline{X}_{S_A^\lambda}$. Since $\underline{X}_{S_A^\lambda} \subseteq X$, we have $x_i \in X$. According to Definition 6, we know that for each $x_j \in S_A^\lambda(x_i)$, $x_j \in X$, which means $m_{ij} = m_{ji} = 1$ and $g_j = 1$. That is, $m_{ij} = m_{ij} g_j$. According to Corollary 3 and Definition 12, $h_i = \frac{\sum_{j=1}^n m_{ij} g_j}{\sum_{j=1}^n m_{ij}} = 1$ and $h_i' = 1$. Hence, for each $i \in \{1, 2, \cdots, n\}$, $g_i' \le h_i'$. That is, $G\left(\underline{X}_{S_A^\lambda}\right) \le H^{[1,1]}(X)$.

$\Leftarrow$: For each $i \in \{1, 2, \cdots, n\}$, if $h_i' = 1$, then $h_i = 1$, which means $\frac{\sum_{j=1}^n m_{ij} g_j}{\sum_{j=1}^n m_{ij}} = 1$. That is $\sum_{j=1}^n m_{ij} = \sum_{j=1}^n m_{ij} g_j$. Then for each $j \in \{1, 2, \cdots, n\}$, if $m_{ij} = 1$, then $g_j = 1$, that means $x_j \in X$. Hence we have $S_A^\lambda(x_i) \subseteq X$, that is $x_i \in \underline{X}_{S_A^\lambda}$ and $g_i' = 1$. Thus, for each $i \in \{1, 2, \cdots, n\}$, $g_i' \ge h_i'$. Therefore, $G\left(\underline{X}_{S_A^\lambda}\right) \ge H^{[1,1]}(X)$.

Therefore, for each $i \in \{1, 2, \cdots, n\}$, we have $g_i' = h_i'$ and $G\left(\underline{X}_{S_A^\lambda}\right) = H^{[1,1]}(X)$.

(2) The proof is similar to that of (1). □

According to Definition 7 and Theorem 1, the positive region $POS_{S_A^\lambda}(X)$, the boundary region $BND_{S_A^\lambda}(X)$, the negative region $NEG_{S_A^\lambda}(X)$ can be generated from the cut matrices, repectively.

(1) The n-column vector $G\left(POS_{S_A^\lambda}(X)\right)$ of the positive region:

$$G\left(POS_{S_A^\lambda}(X)\right) = H^{[1,1]}(X) \tag{16}$$

Table 2
The characteristic matrices of the positive, boundary and negative region on Table 1.

$H(X)$	$G\left(\underline{X}_{S^{0.5}_{A_1}}\right) = H^{[1,1]}(X)$	$G\left(\overline{X}_{S^{0.5}_{A_1}}\right) = H^{(0,1]}(X)$	$G\left(POS^{(0.7,0.3)}_{S^{0.5}_{A_1}}(X)\right) = H^{[0.7,1]}(X)$	$G\left(BND^{(0.7,0.3)}_{S^{0.5}_{A_1}}(X)\right) = H^{(0.7,0.3)}(X)$	$G\left(NEG^{[0,0.3]}_{S^{0.5}_{A_1}}(X)\right) = H^{[0,0.3]}(X)$
$\begin{bmatrix} 1 \\ 1/2 \\ 0 \\ 1/2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(2) The n-column vector $G\left(BND_{S^\lambda_A}(X)\right)$ of the boundary region:

$$G\left(BND_{S^\lambda_A}(X)\right) = H^{(0,1)}(X) \tag{17}$$

(3) The n-column vector $G\left(NEG_{S^\lambda_A}(X)\right)$ of the negative region:

$$G\left(NEG_{S^\lambda_A}(X)\right) = H^{[0,0]}(X) \tag{18}$$

Corollary 4. *Given a pair of threshold (α, β), the (α, β)-probabilistic positive, boundary and negative region can be generated from the cut matrices, repectively.*

(1) The n-column vector $G\left(POS^{(\alpha,\beta)}_{S^\lambda_A}(X)\right)$ of the positive region:

$$G\left(POS^{(\alpha,\beta)}_{S^\lambda_A}(X)\right) = H^{[\alpha,1]}(X) \tag{19}$$

(2) The n-column vector $G\left(BND^{(\alpha,\beta)}_{S^\lambda_A}(X)\right)$ of the boundary region:

$$G\left(BND^{(\alpha,\beta)}_{S^\lambda_A}(X)\right) = H^{(\alpha,\beta)}(X) \tag{20}$$

(3) The n-column vector $G\left(NEG^{(\alpha,\beta)}_{S^\lambda_A}(X)\right)$ of the negative region:

$$G\left(NEG^{(\alpha,\beta)}_{S^\lambda_A}(X)\right) = H^{[0,\beta]}(X) \tag{21}$$

where $0 \leq \beta < \alpha \leq 1$.

Example 5. (Continued from Example 4). In the IvIS given by Table 1, $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, $X = \{x_1, x_4, x_6\}$, $\lambda = 0.5$, $A_1 = \{a_1, a_2, a_5\}$, given a pair of threshold $(\alpha, \beta) = (0.7, 0.3)$, we can obtain the positive, boundary and negative region as follows (Table 2).

4. Matrices updating approaches oriented dynamic IvIS

In order to reduce the computational cost of updating the positive region $POS_{S^\lambda_A}(X)$, boundary region $BND_{S^\lambda_A}(X)$ and negative region $NEG_{S^\lambda_A}(X)$ in dynamic IvIS, we propose incremental matrix updating approaches oriented dynamic IvIS in this section. In general, there are three situations about dynamic IvIS:

(1) variation of the attribute set
(2) variation of the attribute value
(3) variation of the object set.

In terms of traditional algorithm, the key step is to construct an evaluation function and a pair of threshold oriented dynamic IvIs. Considering the 3WD method in Probabilistic Rough Set (PRS), let $Pr\left(X \mid S^\lambda_A(x_i)\right) = \frac{|S^\lambda_A(x_i) \cap X|}{|S^\lambda_A(x_i)|}$ be the evaluation function. The detailed step of traditional algorithm is as shown in Algorithm 1.

Algorithm 1: Traditional algorithm for computing the positive, boundary and negative region of X in term of variation of IvIS (TA-PBN-IS).

input : An interval-valued information system with the attribute set A, a decision class X, a real number $\lambda \in [0.5, 1]$ and a pair of threshold (α, β)
output: the positive region $POS_{S_A^\lambda}(X)$, boundary region $BND_{S_A^\lambda}(X)$ and negative region $NEG_{S_A^\lambda}(X)$

1 **begin**
2   **for** *i,j=1* **to** n **do**
3   **for** *k=1* **to** m **do** /* According to Definition 5 */
4       $S_{ij}^k = \frac{l(f(x_i,a_k)\cap f(x_j,a_k))}{l((fx_i,a_k)\cup f(x_j,a_k))}$;
5   **if** $S_{ij}^k \geq \lambda$ **then** /* According to Definition 6 */
6         $(x_i, x_j) \in S_A^\lambda$;
7   **else**
8         $(x_i, x_j) \notin S_A^\lambda$;
9   **end**
10   **end**
11   **end**
12   **for** *i=1* **to** n **do**
13   **if** $\frac{|S_A^\lambda(x_i)\cap X|}{|S_A^\lambda(x_i)|} \geq \alpha$ **then** /* Compute $POS_{S_A^\lambda}(X)$, $BND_{S_A^\lambda}(X)$ and $NEG_{S_A^\lambda}(X)$ */
14       $x_i \in POS_{S_A^\lambda}(X)$;
15   **else if** $\beta < \frac{|S_A^\lambda(x_i)\cap X|}{|S_A^\lambda(x_i)|} < \alpha$ **then**
16       $x_i \in BND_{S_A^\lambda}(X)$;
17   **else**
18       $x_i \in NEG_{S_A^\lambda}(X)$;
19   **end**
20   **end**
21 **end**
Data: updating the attribute set A; /* variation of A */
22   updating the attribute value set V; /* variation of V */
23   updating the object set U; /* variation of U */
24 **repeat begin-end until** ***Data*** *is not updating*;
output: the positive region $POS_{S_A^\lambda}(X)$, the boundary region $BND_{S_A^\lambda}(X)$, the negative region $NEG_{S_A^\lambda}(X)$

Algorithm 1 is the traditional algorithm to compute the positive, boundary and negative region of X in dynamic IvIS (TA-PBN-IS). The step $2 \to 11$ is to compute similarity degree S_{ij}^k and λ-similarity relation S_A^λ. The step $12 \to 20$ is to compute the positive, boundary and negative region of X. The step $21 \to 24$ is to repeat step $12 \to 20$ while updating IvIS. The total time complexity of MA-PBN-IS is shown in Theorem 2.

Theorem 2. *In TA-PBN-IS, time complexity of step $2 \to 11$ is $O(n^2)$. Time complexity of step $12 \to 20$ is $O(n)$. Time complexity of step $21 \to 24$ is $O(n)$. So, the total time complexity of TA-PBN-IS is $O(n^3)$.*

Proof. Time complexity of algorithm is $T(n) = O(f(n))$, where $f(n)$ represents the execution time of algorithm. For step $2 \to 11$, it is easy to verify that $T_1(n) = n * (n-1) * m$. For step $12 \to 20$, $T_2(n) = n$. Then $T(n) = T_1(n) + T_2(n) = m(n^2 - n) + n$. Since m and n are of the same order, $T(n) = n^3 + n^2 - n$. Let $f(n) = n^3$, there exist real numbers $c = 2$ and $N = 1$ satisfying that when $n \geq N$, $0 \leq T(n) \leq 2 * f(n)$. Total time complexity of TA-PBN-IS is $O\left(n^3\right)$. □

In terms of matrices computing algorithm, under the situation of variation of the attribute set and variation of the value of the attribute set, the key step is to updating the relation matrix $M_{n\times n}^{S_A^\lambda} = \left(m_{ij}\right)_{n\times n}$ and the diagonal matrix $\Lambda_{n\times n}^{S_A^\lambda}$, respectively. While under the situation of variation of the object set, the key step is to updating the characteristic vector $G(X) = (g_1, g_2, \ldots, g_n)^T$. Several algorithms under different situations are introduced as follows.

4.1. Updating the attribute set

Zhang et al. [63] proposed an incremental approach for updating the approximations of rough sets under the situation of variation of the attribute set in set-valued system, which is also effective in IvIS.

Corollary 5. *Let $A_1, A_2 \subseteq A$ be two subsets of A and $A_1 \cap A_2 = \emptyset$. Suppose $M_{n\times n}^{S_{A_1\cup A_2}^\lambda} = \left(m_{ij}^\uparrow\right)_{n\times n}$ be a relation matrix w.r.t. $S_{A_1\cup A_2}^\lambda$, and $M_{n\times n}^{S_{A_1}^\lambda} = \left(m_{ij}\right)_{n\times n}$ be a relation matrix w.r.t. $S_{A_1}^\lambda$. Then $M_{n\times n}^{S_{A_1\cup A_2}^\lambda} = \left(m_{ij}^\uparrow\right)_{n\times n}$ can be updated as follows when adding A_2 into A_1.*

$$m_{ij}^\uparrow = \begin{cases} 0, & m_{ij} = 0 \vee m_{ij} = 1 \wedge (x_i, x_j) \notin S_{A_2}^\lambda \\ 1, & m_{ij} = 1 \wedge (x_i, x_j) \in S_{A_2}^\lambda \end{cases} \tag{22}$$

Proof. If $m_{ij}=0$, which means there exists at least an attribute $a_k(1\le k\le m)\in A_1$ making $(x_i,x_j)\notin S^{\lambda}_{A_1}$, then according to Definition 9, m_{ij} remains unchanged by adding A_2 to A_1, that is $m^{\uparrow}_{ij}=0$. If $m_{ij}=1$, that means for each attribute $a_k(1\le k\le m)\in A_1$, $(x_i,x_j)\in S^{\lambda}_{A_1}$ holds. Then if $(x_i,x_j)\in S^{\lambda}_{A_2}$, that is for each $a_k(1\le k\le m)\in A_1\cup A_2$, we have $(x_i,x_j)\in S^{\lambda}_{A_1\cup A_2}$. According to Definition 9, $m^{\uparrow}_{ij}=1$. Otherwise, $m^{\uparrow}_{ij}=0$. □

Example 6. (Continued from Example 5). In the IvIS given by Table 1, $U=\{x_1,x_2,\ldots,x_9\}$, $\lambda=0.5$, $A_1=\{a_1,a_2,a_5\}$, $A_2=\{a_3,a_4\}$. The relation matrix $M^{S^{\lambda}_{A_1\cup A_2}}_{n\times n}=(m_{ij})_{n\times n}$ can be described as follows.

$$M^{S^{0.5}_{A_1\cup A_2}}_{9\times 9}=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{pmatrix}$$

Corollary 6. *Let $A_2\subset A_1\subseteq A$. Suppose $M^{S^{\lambda}_{A_1-A_2}}_{n\times n}=\left(m^{\downarrow}_{ij}\right)_{n\times n}$ be a relation matrix w.r.t. $S^{\lambda}_{A_1-A_2}$ and $M^{S^{\lambda}_{A_1}}_{n\times n}=(m_{ij})_{n\times n}$ be a relation matrix w.r.t. $S^{\lambda}_{A_1}$. Then $M^{S^{\lambda}_{A_1-A_2}}_{n\times n}=\left(m^{\downarrow}_{ij}\right)_{n\times n}$ can be updated as follows when deleting A_2 from A_1.*

$$m^{\downarrow}_{ij}=\begin{cases}0, & m_{ij}=0\wedge(x_i,x_j)\notin S^{\lambda}_{A_1-A_2}\\ 1, & m_{ij}=1\vee m_{ij}=0\wedge(x_i,x_j)\in S^{\lambda}_{A_1-A_2}\end{cases}\tag{23}$$

Proof. The proof is similar to the Proof of Corollary 5. □

Matrix computing algorithm under the situation of variation of attribute set is as shown in Algorithm 2 and Algorithm 3. Algorithm 2 is matrix computing algorithm for computing the positive, boundary and negative region of X in term of adding an attribute set (MA-PBN-AA) while Algorithm 3 is matrix computing algorithm for computing the positive, boundary and negative region of X in term of deleting an attribute set (MA-PBN-DA). In MA-PBN-AA, step $5\rightarrow 9$ computes characteristic vector $G(X)$. Step $10\rightarrow 17$ is to update relation matrix $M^{S^{\lambda}_{A_1\cup A_2}}_{n\times n}$. The total time complexity of MA-PBN-AA is shown in Theorem 3. Obviously, the total time complexity of MA-PBN-DA is the same as that of MA-PBN-AA. Compared with TA-PBN-IS, MA-PBN-AA and MA-PBN-DA update the relation matrix based on the original ones, which can achieve identical results with less time.

Theorem 3. *In MA-PBN-AA, time complexity of step $3\rightarrow 9$ is $O(n)$. Time complexity of step $3\rightarrow 20$ is $O(n^2)$. Time complexity of step $21\rightarrow 24$ is $O(n)$. The total time complexity of TA-PBN-AA is $O(n^2)$.*

Proof. For step $3\rightarrow 20$, $T(n)=1+2+\cdots+(n-1)=\frac{n(n-1)}{2}$. Let $f(n)=n^2$, there exist real numbers $c=1$ and $N=1$ satisfying that when $n\ge N$, $0\le T(n)\le f(n)$. So, total time complexity of TA-PBN-AA is $O\left(n^2\right)$. □

Obviously, the total time complexity of MA-PBN-DA also is $O(n^2)$. Compared with TA-PBN-IS, MA-PBN-AA and MA-PBN-DA update the relation matrix based on the original ones, which can achieve identical results with less time.

4.2. Updating the attribute value

Corollary 7. *Given an $IvIS^t=\left(U^t,A^t,V^t,f^t\right)$ for time t. Suppose the value of x_i under the attribute a_k be changed in time $t+1$, which means $f^{t+1}(x_i,a_k)\neq f^t(x_i,a_k)$. Then the relation matrix $M^{S^{\lambda}_{A}}_{n\times n}=(m_{ij})^{t+1}_{n\times n}$ can be updated as follows.*

$$m^{t+1}_{ij}=\begin{cases}0, & \left(m^t_{ij}=0\wedge(S^k_{ij})^t\ge\lambda\right)\vee\left(m^t_{ij}=1\wedge(S^k_{ij})^{t+1}<\lambda\right)\\ 1, & \left(m^t_{ij}=0\wedge(x_i,x_j)^{t+1}\in S^{\lambda}_{A}\right)\vee\left(m^t_{ij}=1\wedge(S^k_{ij})^{t+1}\ge\lambda\right)\end{cases}\tag{24}$$

Algorithm 2: Matrix computing algorithm for computing the positive, boundary and negative region of X in term of adding an attribute set (MA-PBN-AA).

input : An interval-valued information system with the attribute set A, a decision class X and a pair of threshold (α, β); Adding an attribute A_2
output: the positive region $POS_{S^{\lambda}_{A_1+A_2}}(X)$, boundary region $BND_{S^{\lambda}_{A_1+A_2}}(X)$ and negative region $NEG_{S^{\lambda}_{A_1+A_2}}(X)$

begin
  $A_1 + A_2 \rightarrow A_1$;
  for *i=1* **to** n **do**
    $m^{\uparrow}_{ii} = 1$;
    if $x_i \in X$ **then** /* According to Definition 9 */
      $g_i = 1$;
    else
      $g_i = 0$;
    end
    for *j=i+1* **to** n **do** /* According to Corollary 6 */
      if $m_{ij} == 0$ **then**
        $m^{\uparrow}_{ij} = 0$;
      else if $(x_i, x_j) \in S^{\lambda}_{A_2}$ **then**
        $m^{\uparrow}_{ij} = 1$;
      else
        $m^{\uparrow}_{ij} = 0$;
      end
      $\lambda_i = \lambda_i + m^{\uparrow}_{ij}$; /* Update λ_i */
    end
  end
  Compute $\Lambda^{S^{\lambda}_{A_1+A_2}}_{n\times n} = diag\left(\frac{1}{\sum_{j=1}^{n} m_{1j}}, \frac{1}{\sum_{j=1}^{n} m_{2j}}, \cdots, \frac{1}{\sum_{j=1}^{n} m_{nj}}\right)$
  Compute $H(X) = \Lambda^{S^{\lambda}_{A_1+A_2}}_{n\times n} \bullet \left(M^{S^{\lambda}_{A_1+A_2}}_{n\times n} \bullet G(X)\right)$ /* Update $H(X)$ */
end

output: $G\left(POS^{(\alpha,\beta)}_{S^{\lambda}_{A_1+A_2}}(X)\right) = H^{[\alpha,1]}(X)$, $G\left(BND^{(\alpha,\beta)}_{S^{\lambda}_{A_1+A_2}}(X)\right) = H^{(\alpha,\beta)}(X)$, $G\left(NEG^{(\alpha,\beta)}_{S^{\lambda}_{A_1+A_2}}(X)\right) = H^{[0,\beta]}(X)$

Algorithm 3: Matrix computing algorithm for computing the positive, boundary and negative region of X in term of adding an attribute set (MA-PBN-DA).

input : An interval-valued information system with the attribute set A, a decision class X and a pair of threshold (α, β); Deleting an attribute A_2
output: the positive region $POS_{S^{\lambda}_{A_1-A_2}}(X)$, boundary region $BND_{S^{\lambda}_{A_1-A_2}}(X)$ and negative region $NEG_{S^{\lambda}_{A_1-A_2}}(X)$

begin
  $A_1 - A_2 \rightarrow A_1$;
  for *i=1* **to** n **do**
    $m^{\downarrow}_{ii} = 1$;
    if $x_i \in X$ **then** /* According to Definition 9 */
      $g_i = 1$;
    else
      $g_i = 0$;
    end
    for *j=i+1* **to** n **do** /* According to Corollary 6 */
      if $m_{ij} == 1$ **then**
        $m^{\downarrow}_{ij} = 1$;
      else if $(x_i, x_j) \in S^{\lambda}_{A_2}$ **then**
        $m^{\downarrow}_{ij} = 1$;
      else
        $m^{\downarrow}_{ij} = 0$;
      end
      $\lambda_i = \lambda_i + m^{\downarrow}_{ij}$; /* Update λ_i */
    end
  end
  Compute $\Lambda^{S^{\lambda}_{A_1-A_2}}_{n\times n} = diag\left(\frac{1}{\sum_{j=1}^{n} m_{1j}}, \frac{1}{\sum_{j=1}^{n} m_{2j}}, \cdots, \frac{1}{\sum_{j=1}^{n} m_{nj}}\right)$
  Compute $H(X) = \Lambda^{S^{\lambda}_{A_1-A_2}}_{n\times n} \bullet \left(M^{S^{\lambda}_{A_1-A_2}}_{n\times n} \bullet G(X)\right)$ /* Update $H(X)$ */
end

output: $G\left(POS^{(\alpha,\beta)}_{S^{\lambda}_{A_1-A_2}}(X)\right) = H^{[\alpha,1]}(X)$,$G\left(BND^{(\alpha,\beta)}_{S^{\lambda}_{A_1-A_2}}(X)\right) = H^{(\alpha,\beta)}(X)$,$G\left(NEG^{(\alpha,\beta)}_{S^{\lambda}_{A_1-A_2}}(X)\right) = H^{[0,\beta]}(X)$

Algorithm 4: Matrix computing algorithm for computing the positive, boundary and negative region of X in term of variation of value of attribute set (MA-PBN-VA).

input : An interval-valued information system with the $f^{t+1}(x_i, a_k)$ changed at time $t+1$, a decision class X and a pair of threshold (α, β)
output: the positive region $POS_{S_A^\lambda}(X)$, boundary region $BND_{S_A^\lambda}(X)$ and negative region $NEG_{S_A^\lambda}(X)$

1 **begin**
2 $f^t(x_i, a_k) \to f^{t+1}(x_i, a_k)$;
3 **for** *i=1* **to** n **do**
4 $m_{ii}^{t+1} = 1$;
5 **if** $x_i \in X$ **then** /* According to Definition 9 */
6 $g_i = 1$;
7 **else**
8 $g_i = 0$;
9 **end**
10 **for** *j=i+1* **to** n **do** /* According to Corollary 6 */
11 **if** $m_{ij}^{t+1} == 1$ **then**
12 **if** $(S_{ij}^k)^{t+1} < \lambda$ **then**
13 $m_{ij}^{t+1} = 0$;
14 **else**
15 $m_{ij}^{t+1} = 1$;
16 **end**
17 **else**
18 **if** $(x_i, x_j)^{t+1} \in S_A^\lambda$ **then**
19 $m_{ij}^{t+1} = 1$;
20 **else**
21 $m_{ij}^{t+1} = 0$;
22 **end**
23 **end**
24 $\lambda_i = \lambda_i + m_{ij}^{t+1}$; /* Update λ_i */
25 **end**
26 **end**
27 **Compute** $\Lambda_{n\times n}^{S_A^\lambda} = diag\left(\frac{1}{\sum_{j=1}^n m_{1j}}, \frac{1}{\sum_{j=1}^n m_{2j}}, \cdots, \frac{1}{\sum_{j=1}^n m_{nj}}\right)$
28 **Compute** $H(X) = \Lambda_{n\times n}^{S_A^\lambda} \bullet \left(M_{n\times n}^{S_A^\lambda} \bullet G(X)\right)$ /* Update $H(X)$ */
29 **end**

output: $G\left(POS_{S_A^\lambda}^{(\alpha,\beta)}(X)\right) = H^{[\alpha,1]}(X)$, $G\left(BND_{S_A^\lambda}^{(\alpha,\beta)}(X)\right) = H^{(\alpha,\beta)}(X)$, $G\left(NEG_{S_A^\lambda}^{(\alpha,\beta)}(X)\right) = H^{[0,\beta]}(X)$

Proof. Suppose $m_{ij}^t = 0$. If $(S_{ij}^k)^t \geq \lambda$, which means there exists an attribute $a_l (l \neq k) \in A$ satisfying $(S_{ij}^l)^t < \lambda$, then $m_{ij}^{t+1} = 0$. If $(S_{ij}^k)^t < \lambda$, according to Definition 6, $m_{ij}^{t+1} = 1$ holds only if $(x_i, x_j)^{t+1} \in S_A^\lambda$; Suppose $m_{ij}^t = 1$, which means for each $a_l \in A$, $(S_{ij}^l)^t \geq \lambda$ holds. Then if $(S_{ij}^k)^{t+1} < \lambda$, $m_{ij}^{t+1} = 0$, else $m_{ij}^{t+1} = 1$. □

Matrix computing algorithm under the situation of variation of attribute value set is as shown in Algorithm 4. In MA-PBN-VA, step $3 \to 9$ computes characteristic vector $G(X)$. The step $10 \to 26$ is to update relation matrix $M_{n\times n}^{S_A^\lambda}$. The total time complexity of MA-PBN-VA is shown in Theorem 4. MA-PBN-VA also has less time complexity compared with TA-PBN-IS.

Theorem 4. *In TA-PBN-VA, time complexity of step $3 \to 9$ is $O(n)$. Time complexity of step $3 \to 26$ is $O(n^2)$. So, the total time complexity of TA-PBN-IS is $O(n^2)$.*

Proof. The proof is similar to the Proof of Theorem 3. □

4.3. Updating the object set

Corollary 8. *Given an $IvIS^t = (U^t, A^t, V^t, f^t)$ for time t, $X \subseteq U^t$, $G^t(X)$ and $G^{t+1}(X)$ are characteristic vector at time t and $t+1$, respectively. Suppose a new object set $\Delta U = \{x_{n+1}, x_{n+2}, \cdots, x_{n+m}\}$ adding to U^t at time $t+1$, that is $U^{t+1} = U^t \cup \Delta U$, $X^{t+1} = X^t \cup \Delta X$, $\Delta X \subseteq \Delta U$. Then $G^{t+1}(X) = \left(g_1^{t+1}, g_2^{t+1}, \cdots, g_{n+1}^{t+1}, \cdots, g_{n+m}^{t+1}\right)^T$ can be updated as follows.*

$$g_i^{t+1} = \begin{cases} g_i^t, & 1 \leq i \leq n \\ 1, & x_i \in \Delta X \wedge n+1 \leq i \leq n+m \\ 0, & x_i \notin \Delta X \wedge n+1 \leq i \leq n+m \end{cases} \tag{25}$$

Algorithm 5: Matrix computing algorithm for computing the positive, boundary and negative region of X in term of variation of object set (MA-PBN-VO).

input : An interval-valued information system at time $t+1$ with a new object set $\Delta U = \{x_{n+1}, x_{n+2}, \cdots, x_{n+m}\}$, a decision class X and a pair of threshold (α, β)
output: the positive region $POS_{S_A^\lambda}(X)$, boundary region $BND_{S_A^\lambda}(X)$ and negative region $NEG_{S_A^\lambda}(X)$

1 **begin**
2 $\quad U^t \cup \Delta U \to U^t$; $X^t \cup \Delta X \to X^t$;
3 $\quad$ **for** *i=1* **to** *n+m* **do**
4 $\quad\quad m_{ii}^{t+1} = 1$;
5 $\quad\quad$ **if** $1 \le i \le n$ **then** /* According to Definition 9 */
6 $\quad\quad\quad g_i^{t+1} = g_i^t$;
7 $\quad\quad\quad$ **for** *j=i+1* **to** *n+m* **do** /* According to Corollary 6 */
8 $\quad\quad\quad\quad$ **if** $(x_i, x_j)^{t+1} \in S_A^\lambda$ **then**
9 $\quad\quad\quad\quad\quad m_{ij}^{t+1} = 1$;
10 $\quad\quad\quad\quad$ **else**
11 $\quad\quad\quad\quad\quad m_{ij}^{t+1} = 0$;
12 $\quad\quad\quad\quad$ **end**
13 $\quad\quad\quad$ **end**
14 $\quad\quad$ **else if** $x_i \in \Delta X$ **then**
15 $\quad\quad\quad g_i^{t+1} = 1$;
16 $\quad\quad$ **else**
17 $\quad\quad\quad g_i^{t+1} = 0$;
18 $\quad\quad$ **end**
19 $\quad\quad \lambda_i = \lambda_i + m_{ij}^{t+1}$; /* Update λ_i */
20 $\quad$ **end**
21 $\quad$ **Compute** $\Lambda_{n\times n}^{S_A^\lambda} = diag\left(\frac{1}{\sum_{j=1}^n m_{1j}}, \frac{1}{\sum_{j=1}^n m_{2j}}, \cdots, \frac{1}{\sum_{j=1}^n m_{nj}}\right)$
22 $\quad$ **Compute** $H(X) = \Lambda_{n\times n}^{S_A^\lambda} \bullet \left(M_{n\times n}^{S_A^\lambda} \bullet G(X)\right)$ /* Update $H(X)$ */
23 **end**

output: $G\left(POS_{S_A^\lambda}^{(\alpha,\beta)}(X)\right) = H^{[\alpha,1]}(X)$,$G\left(BND_{S_A^\lambda}^{(\alpha,\beta)}(X)\right) = H^{(\alpha,\beta)}(X)$,$G\left(NEG_{S_A^\lambda}^{(\alpha,\beta)}(X)\right) = H^{[0,\beta]}(X)$

Proof. Let $U^{t+1} = U^t \cup \Delta U$. Obviously, when $1 \le i \le n$, characteristic vector $G^{t+1}(X)$ is unchanged. According to Definition 8, when $n+1 \le i \le n+m$, if $x_i \in \Delta X$, $g_i^{t+1} = 1$. Otherwise, $g_i^{t+1} = 0$ $\square$

Matrix computing algorithm under the situation of variation of object set is shown in Algorithm 5. In MA-PBN-VA, step $5 \to 18$ updates characteristic vector $G(X)^{t+1}$. The step $7 \to 13$ is to compute relation matrix $M_{(n+m)\times(n+m)}^{S_A^\lambda}$. The total time complexity of MA-PBN-VO is shown in Theorem 5.

Theorem 5. *In TA-PBN-VO, time complexity of step* $7 \to 13$ *is* $O(n)$. *Time complexity of step* $3 \to 20$ *is* $O(n^2)$. *So, the total time complexity of TA-PBN-VO is* $O(n^2)$.

Proof. The proof is similar to the Proof of Theorem 3. $\square$

5. Experimental evaluations

In this section, in order to test performance of TA-PBN-IS, MA-PBN-AA, MA-PBN-VA and MA-PBN-VO, we download six real number data sets from the University of California at Irvine (UCI) machine learning data repository and compare computational time of algorithms on UCI data sets. The description of data sets is shown in Table 3. In particular, as an important input parameter of the algorithm, X is generated by the decision attribute in IvIS generally. Considering that some datasets in Table 3 do not contain decision attribute and in order to illustrate the effectiveness of incremental algorithms for any decision class, X is randomly generated in experiment. Firstly, we need to transform real number data sets into interval-valued data sets. Refer to the statistical method mentioned in [16], we increase by 10% of the real number as the upper bound of interval-valued number and decrease by 10% as the lower bound, respectively. Then we compare the computational time of TA-PBN-IS, MA-PBN-AA, MA-PBN-VA and MA-PBN-VO oriented dynamic data sets. The computations are conducted on a PC with Windows 10 and Intel(R) Core(TM) i7-9750H CPU @ 2.60 GHz. Algorithms run in MATLAB 2018b.

Table 3
A description of data sets.

Serial number	Data sets	Abbreviation	Instance	Attributes
1	Wine	Wine	178	13
2	Ecoli	Ecoli	336	8
3	Abalone	Abalone	4177	8
4	Waveform Database Generator (Version 2)	WDGV2	5000	40
5	Gait Classification	GC	48	321
6	ForestFire	FF	517	13

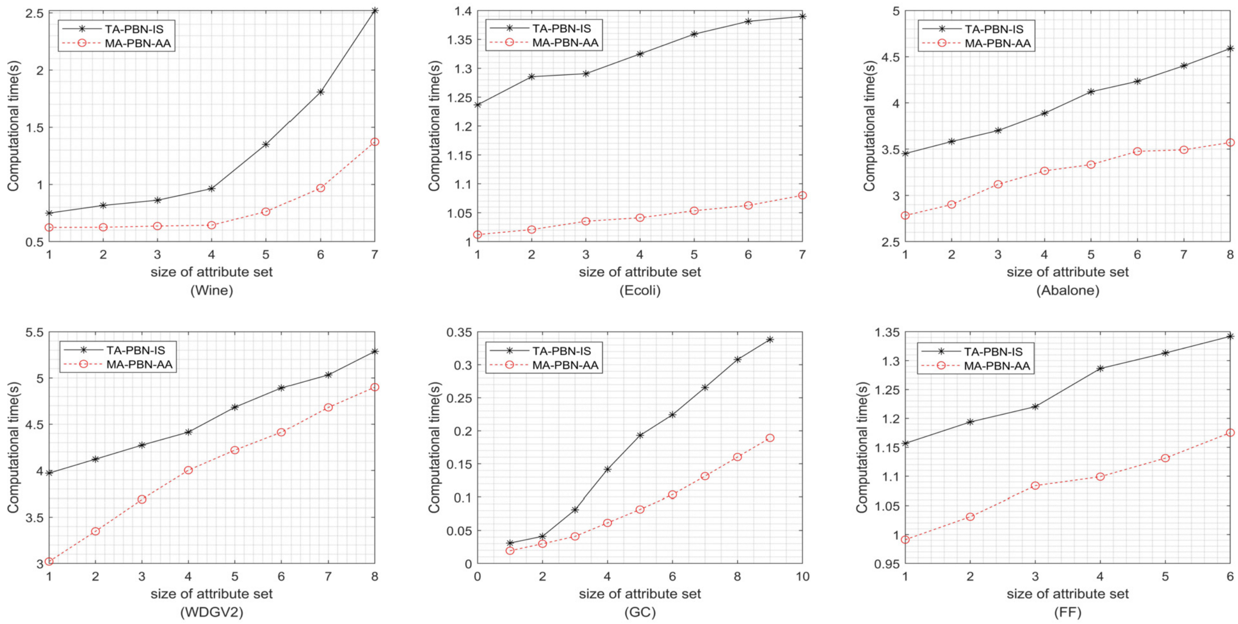

Fig. 1. The comparison of computation time between Algorithm TA-PBN-IS and MA-PBN-AA versus the size of data sets.

5.1. Comparison between TA-PBN-IS and MA-PBN-AA

Under the situation of updating the attribute set, we divide each data sets in Table 3 into equal parts according to the number of attributes. Then we gradually merge these sub datasets and compare computational time between TA-PBN-IS and MA-PBN-AA. The comparative result is shown in Fig. 1.

From Fig. 1, we can see clearly that computational time of Algorithm TA-PBN-IS and MA-PBN-AA increase with the increasing of size of attribute sets. Besides, we can draw such a conclusion that MA-PBN-AA is always faster than TA-PBN-IS on every data set in Table 3.

5.2. Comparison between TA-PBN-IS and MA-PBN-VA

Under the situation of updating the value of attribute set, we gradually update the attribute value sets in Table 3 with the ratio from 0% to 50%. Then we compare computational time between TA-PBN-IS and MA-PBN-VA. The comparative result is shown in Fig. 2.

From Fig. 2, we can see that computational time of TA-PBN-IS or MA-PBN-AA differs from the updating ratio of attribute sets. And it is also clearly that MA-PBN-VA is always faster than TA-PBN-IS on every data set in Table 3.

5.3. Comparison between TA-PBN-IS and MA-PBN-VO

Under the situation of updating the object set, we divide each data sets in Table 3 into equal parts according to the number of objects. In the same way, we gradually merge these sub datasets and compare computational time between TA-PBN-IS and MA-PBN-VO. The comparative result is shown in Fig. 3.

From Fig. 3, it is also clear that computational time of TA-PBN-IS and MA-PBN-VO increase with the increasing of size of object sets. Similarly, MA-PBN-VO is always faster than TA-PBN-IS on every data set in Table 3.

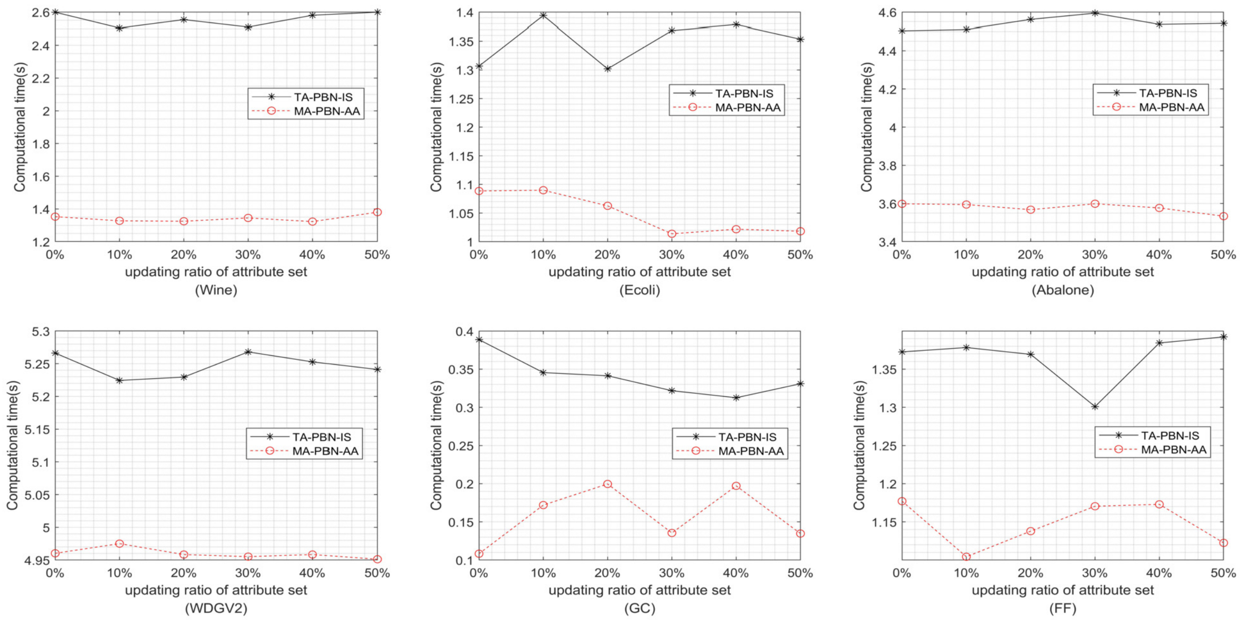

Fig. 2. The comparison of computation time between Algorithm TA-PBN-IS and MA-PBN-VA versus the size of data sets.

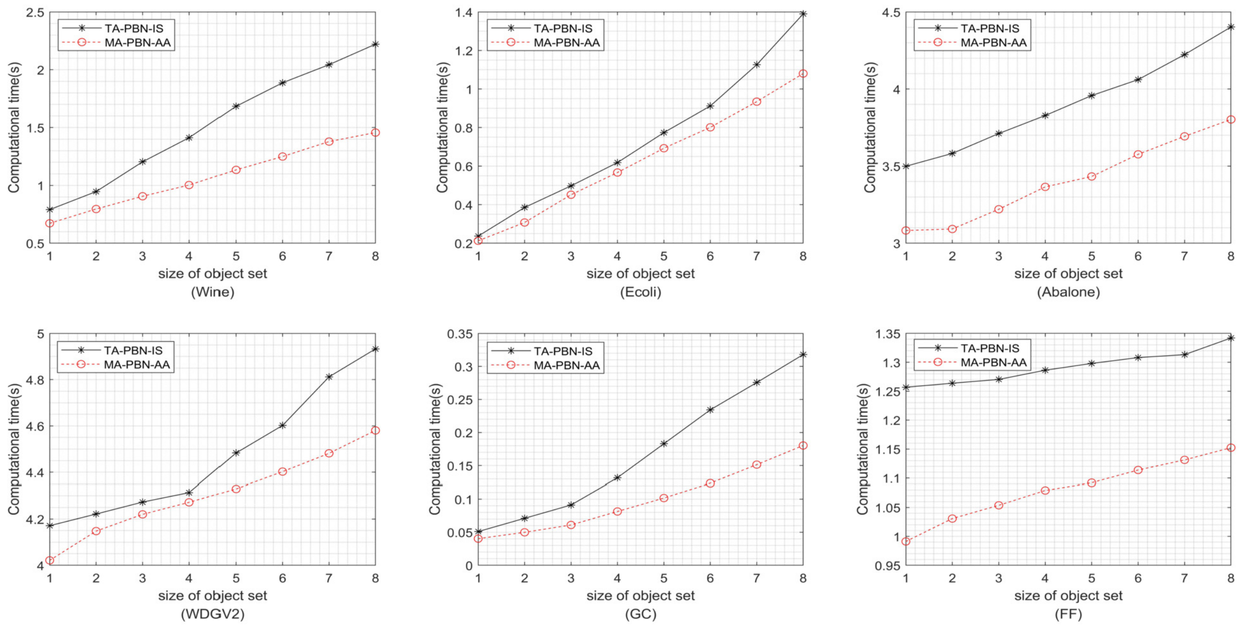

Fig. 3. The comparison of computation time between Algorithm TA-PBN-IS and MA-PBN-VO versus the size of data sets.

6. Conclusion

IvIS can effectively reflect the uncertain information in real life. In fact, in the real scene, the uncertain information is constantly changing, and these changes can be comprehensively reflected in the dynamic IvIS. There are three situations in dynamic IvIS: updating of attribute sets, updating of attribute value and updating of object sets. In this paper, a matrix computational framework based on λ-similarity relation is proposed. Then corresponding algorithms are established respectively to calculate the positive, boundary and negative region on IvIS under three situations mentioned above. Each algorithm is compared with the traditional algorithm on six data sets from UCI. The results show that the proposed algorithms run faster and are more effective than the traditional algorithm. Our future work will focus on three-way decisions methods on mixed data sets in dynamic data sets.

CRediT authorship contribution statement

Ji Shi: Conceptualization, Formal analysis, Investigation, Methodology, Software, Writing – original draft. **Zhongying Suo:** Data curation, Funding acquisition, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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